

# MORE ABOUT ADDITIVE REPRESENTATION FUNCTIONS FOR INTEGERS

LABIB HADDAD

In a recent note [3], posted on arXiv, (16 Jul 2015), KISS and SÁNDOR “improve a result of Haddad and Helou about the Erdős-Turán conjecture”. Can this improvement be still improved ? We try to answer that question.

Here are a few definitions and notations, stripped down as much as can be, for simplicity sake.

Let  $X$  be any subset of  $\mathbb{N} = \{0, 1, \dots\}$ . A  **$X$ -representation** of an integer  $n$  is any ordered couple  $(x, y) \in X \times X$  such that  $n = x + y$ . Let  $r(X, n)$  be the number of all those  $X$ -representations of  $n$ . The function  $n \mapsto r(X, n)$  is the **representation function** relative to  $X$ . Set  $s(X) = \sup\{r(X, n) : n \in \mathbb{N}\}$ . Now,  $s(X)$  is either an integer or  $\infty$ , according to cases, and it produces, thus, a **dichotomy** among the subsets of  $\mathbb{N}$ . We say that  $X$  is in **the upper class** whenever  $s(X) = \infty$ . Otherwise, say that  $X$  is in **the lower class**.

All along,  $A$  and  $B$  are two given infinite subsets of  $\mathbb{N}$ . We enumerate  $A$  and  $B$  in increasing order:

$$A = \{a_1 < a_2 < \dots\} \quad , \quad B = \{b_1 < b_2 < \dots\},$$

then set

$$A(k) = \{a_1 < a_2 < \dots < a_k\} \quad , \quad B(k) = \{b_1 < b_2 < \dots < b_k\},$$

$$u(k) = s(A(k)) \quad , \quad v(k) = s(B(k)).$$

Clearly enough,  $u(k)$  and  $v(k)$  are monotone, non-decreasing, functions of  $k$ , and  $s(A)$ ,  $s(B)$ , are their respective limits when  $k$  approaches  $\infty$ . Also, set:

$$d(k) = \sup \{|a_i - b_i| : i \leq k\},$$

$$d = \sup\{d(k) : k \geq 1\} = \lim_{k \rightarrow \infty} d(k).$$

Thus,  $d$  is a measure of proximity, or closeness, between the two subsets  $A$  and  $B$ . It seems reasonable to expect that whenever  $A$  and  $B$  are

close enough, in a certain sense, they both belong to the same class, upper or lower. That this is indeed the case has been already noticed a long time ago. More specifically, **if  $d$  is finite, then  $A$  and  $B$  belong, both, to the same class**: See, for example, in [1], Corollary 3.4, page 88, and the inequalities:

$$(0) \quad \frac{s(A)}{4d+1} \leq s(B) \leq (4d+1)s(A).$$

### What if $d$ is infinite ?

Of course, for  $k \geq 1$ , the following more general inequalities still hold:

$$(1) \quad \frac{u(k)}{4d(k)+1} \leq v(k) \leq (4d(k)+1)u(k).$$

**Proof.** For  $m, n \in \mathbb{N}$ , set

$$E(k, m) = \{(i, j) : i, j \leq k, a_i + a_j = m\},$$

$$F(k, n) = \{(i, j) : i, j \leq k, b_i + b_j = n\}.$$

For  $(i, j) \in F(k, n)$ , we have

$$b_i - d(k) \leq a_i \leq b_i + d(k)$$

$$b_j - d(k) \leq a_j \leq b_j + d(k)$$

so that

$$b_i + b_j - 2d(k) \leq a_i + a_j \leq b_i + b_j + 2d(k),$$

$$n - 2d(k) \leq a_i + a_j \leq n + 2d(k).$$

Therefore, each couple  $(i, j) \in F(k, n)$  belongs to one of the sets  $E(k, m)$  for some  $m \in [n - 2d(k), n + 2d(k)]$ . The number of couples in  $F(k, n)$  is  $r(B(k), n)$ . The number of couples in  $E(k, m)$  is  $r(A(k), m) \leq u(k)$ . Taking for  $n$  an integer having the maximum number of  $B(k)$ -representations, i.e.,  $r(B(k), n) = v(k)$ , we thus obtain:

$$v(k) \leq (4d(k)+1)u(k).$$

Exchanging  $A$  and  $B$  obtains the result.  $\square$

Let us introduce a new function:

$$(2) \quad w(A, B) = \sup_{k \geq 1} \frac{u(k)}{4d(k)+1}.$$

{One might as well call  $w$  the *wizard*.} Then, clearly enough, we have

$$(3) \quad w(A, B) \leq s(B) \text{ and } w(A, B) \leq \sup u(k) = s(A).$$

We similarly have, of course,  $w(B, A) \leq s(A) \wedge s(B)$ .

**Scholium.** *Two given subsets  $A$  and  $B$  of  $\mathbb{N}$  are both in the upper class if  $w(A, B)$  is infinite. Indeed, if  $w(A, B)$  is infinite, then so are  $s(A)$  and  $s(B)$ , by (3).*

### A SPECIAL CASE

Take  $A$  to be the set of squares,  $A = \{1, 4, 9, \dots, n^2, \dots\}$ . Consider  $u(k) = s(A(k))$ . It is well-known that  $u(k)$  is unbounded, that is,  $s(A)$  is infinite: Just remember, for instance, Jacobi's formula for the number of representations of an integer as a sum of two squares.

Take any function  $f(k) > 0$  such that  $u(k)/f(k)$  is unbounded. If, for a given subset  $B$  of  $\mathbb{N}$ , we have  $d(k) \leq f(k)$ , then  $B$  is in the upper class.

Otherwise stated: **Whenever we have  $|\mathbf{b}_n - \mathbf{n}^2| \leq \mathbf{f}(\mathbf{k})$ , for each  $n \leq k$ , the subset  $B$  belongs to the upper class.**

One way to choose such a function  $f$  is to take a function  $g(k) > 0$  such that  $\lim_{k \rightarrow \infty} g(k) = 0$ , and let  $f(k) = u(k)g(k)$ . Examples abound.

This is an ample generalization of theorem 2 in [3].

{See [2] for the mentioned result of Haddad and Helou.}

### REFERENCES

1. G. Grekos, L. Haddad, C. Helou, and J. Pihko, *The class of Erdős-Turán sets*, Acta Arith. **117** (2005), 81-105.
2. L. Haddad, C. Helou, *Representations of integers by near quadratic sequences*, Journal of Integer sequences **15** (2012), 12.8.8.
3. Sándor Z. Kiss and Csaba Sándor, *On the maximum values of the additive representation functions*, arXiv:1504.07411v2.